Orbits of particles in noncommutative Schwarzschild spacetime

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Abstract

By considering particles as smeared objects, we investigate the effects of space noncommutativity on the orbits of particles in Schwarzschild spacetime. The effects of space noncommutativity on the value of the precession of the perihelion of particle orbit and deflection of light ray in Schwarzschild geometry are calculated and the stability of circular orbits is discussed.

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1 Introduction

Gedanken experiments that aim at probing spacetime structure at very small distances support the idea that noncommutativity of spacetime is a feature of Planck scale physics. They show that due to gravitational back reaction, one cannot test spacetime at Planck scale. Its description as a smooth manifold becomes therefore a mathematical assumption no more justified by physics. It is then natural to relax this assumption and conceive a more general noncommutative spacetime, where uncertainty relations and discretization naturally arise. Noncommutativity is the central mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables, such as position and momentum. One can just as easily imagine that position measurements might fail to commute and describe this using noncommutativity of the coordinates. The noncommutativity of spacetime can be encoded in the commutator[1-7]

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \tag{1}$$

where θ^{ij} is a real, antisymmetric and constant tensor, which determines the fundamental cell discretization of spacetime much in the same way as the Planck constant $\hbar$ discretizes the phase space. In $d = 4$, by a choice of coordinates, this noncommutativity can be brought to the form

$$\theta^{ij} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & \theta & 0 \\ 0 & -\theta & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}$$

This was motivated by the need to control the divergences showing up in theories such as quantum electrodynamics. This noncommutativity leads to the modification of Heisenberg uncertainty relation in such a way that prevents one from measuring positions to better accuracies than the Planck length. In low energy limit, these quantum gravity effects can be neglected, but in circumstances such as very early universe or in the strong gravitational field one has to consider these effects. The purpose of this paper is to investigate the effects of space noncommutativity on the orbits of a test particle in noncommutative Schwarzschild geometry. The modifications induced by the generalized uncertainty principle on the classical orbits of particles in a central force potential firstly has been considered by Benczik *et al* [8]. The same problem has been considered within noncommutative geometry by Mirza and Dehghani[9]and also by Romero and Vergara[10]. The main consequence of these investigations is the constraint imposed on the minimal observable length and noncommutativity parameter in comparison with observational

data of Mercury. Recently, stability of planetary orbits of particles in noncommutative space has been studied both in central force and Schwarzschild background by Nozari and Akhshabi[11]. Here we are going to look at the Kepler problem in noncommutative Schwarzschild geometry from a different viewpoint. It has been shown that noncommutativity eliminates point-like structures in favor of smeared objects in flat spacetime[12]. The effect of smearing can be mathematically implemented as a substitution rule: position Dirac-delta function can be replaced everywhere with a Gaussian distribution of minimal width $\sqrt{\theta}$. In this framework, the mass density of a static, spherically symmetric, smeared, particle-like gravitational source can be shown by a Gaussian profile[13]. By adapting such a setup, we will find a generalized orbit equation. Then we calculate the modification imposed by space noncommutativity on the value of the precession of the perihelion of Mercury. The effect of space noncommutativity on the deflection of light ray in Schwarzschild geometry is calculated and its numerical value is obtained. Finally the issue of stability of circular orbits is discussed and the condition for a circular orbit to be stable in noncommutative Schwarzschild geometry is obtained.

2 Noncommutative Schwarzschild Spacetime

Commutative Schwarzschild geometry is described by the following line element

$$ds^2 = \Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 - r^2 d\Omega^2, \tag{2}$$

were we have set $G = 1 = c$. We want to consider the effect of space noncommutativity on this line element. Note that it is possible to consider the effects of space noncommutativity on Einstein field equations also. One can argue that it is not necessary to change the Einstein tensor part of the field equations, and that the noncommutative effects can be implemented acting only on the matter source[12,13,14]. Since noncommutativity eliminates point-like objects in favor of smeared objects in flat spacetime, we choose the mass density of a static, spherically symmetric smeared, particle-like gravitational source as[13]

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} \exp(-\frac{r^2}{4\theta}). \tag{3}$$

Solving the Einstein equations with this matter source, one finds the following noncommutative Schwarzschild line element[13]

$$ds^2 = \Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\Big)dt^2 - \Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\Big)^{-1} dr^2 - r^2 d\Omega^2 \tag{4}$$

where $\gamma(3/2, r^2/4\theta)$ is the lower incomplete Gamma function defined as (see appendix)

$$\gamma(3/2, r^2/4\theta) \equiv \int_0^{r^2/4\theta} dt \ t^{1/2} e^{-t} \tag{5}$$

In the limit of $r/\sqrt{\theta} \to \infty$ the classical Schwarzschild metric is obtained. Using noncommutative Schwarzschild line element as (4), in the next section we investigate the effect of space noncommutativity on the orbits of a test particle.

3 Orbits of Particles

Along an affinely parameterized geodesic (timelike , spacelike or null) the scalar quantity $2K = u^\alpha u_\alpha$ is a constant[15]

$$2K = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}. \tag{6}$$

If the proper time or proper distance is chosen for λ, then $2K = \pm 1$ for timelike and spacelike intervals. For a null geodesic $2K = 0$. Using the line element (4) we find for the timelike geodesics

$$2K = \Big(1 - \frac{4M}{r\sqrt{\pi}} \gamma(3/2, r^2/4\theta)\Big)\dot{t}^2 - \Big(1 - \frac{4M}{r\sqrt{\pi}} \gamma(3/2, r^2/4\theta)\Big)^{-1} \dot{r}^2 - r^2 \dot{\vartheta}^2 - r^2 \sin^2(\vartheta) \dot{\varphi}^2 = 1. \tag{7}$$

Using the Euler-Lagrange equation

$$\frac{\partial K}{\partial x^a} - \frac{d}{d\tau}\Big(\frac{\partial K}{\partial x^a}\Big) = 0, \tag{8}$$

for $a = 0,\ 2,\ 3$ we find

$$\frac{d}{d\tau}\Big[\Big(1 - \frac{4M}{r\sqrt{\pi}} \gamma(3/2, r^2/4\theta)\Big)\dot{t}\Big] = 0, \tag{9}$$

$$\frac{d}{d\tau}\Big(r^2 \dot{\vartheta} - r^2 \sin\vartheta \cos\vartheta \dot{\varphi}^2\Big) = 0, \tag{10}$$

and

$$\frac{d}{d\tau}\Big(r^2 sin^2 \vartheta \dot{\varphi}\Big) = 0. \tag{11}$$

These three equations along with equation (7) provide us with the four equations needed for determining the four desired relations, namely

$$t = t(\tau), \qquad r = r(\tau), \qquad \vartheta = \vartheta(\tau), \qquad \varphi = \varphi(\tau).$$

One also could find the trajectory of the test particle projected into a slice $t = const$. To do this end, we consider motion in the equatorial plane, $\vartheta = \frac{\pi}{2}$. From the equation (11) we find

$$r^2\dot{\varphi} = h, \tag{12}$$

where h is constant of integration. Also, integrating equation (9), we get

$$\Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\Big)\dot{t} = k, \tag{13}$$

where k is constant of integration. Substituting (13) into relation (7), we obtain

$$k^2\Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma\Big)^{-1} - \Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma\Big)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = 1. \tag{14}$$

As usual, we define $u \equiv r^{-1}$ which leads to

$$\dot{r} = -h\frac{du}{d\varphi}. \tag{15}$$

Using relations (12), (13) and (15) in equation (14), we find the following differential equation for the orbit of a test particle in noncommutative Schwarzschild geometry

$$\Big(\frac{du}{d\varphi}\Big)^2 + u^2 = \frac{k^2 - 1}{h^2} + \frac{4Mu\gamma}{\sqrt{\pi}h^2} + \frac{4Mu^3\gamma}{\sqrt{\pi}} \tag{16}$$

Differentiating (16) we find the second order orbit equation

$$\frac{d^2u}{d\varphi^2} + u = \frac{2M}{\sqrt{\pi}h^2}\gamma - \frac{M}{2\sqrt{\pi}h^2\theta^{3/2}}\frac{e^{-1/4\theta u^2}}{u^3} + \frac{6Mu^2}{\sqrt{\pi}}\gamma - \frac{M}{2\sqrt{\pi}\theta^{3/2}}\frac{e^{-1/4\theta u^2}}{u}. \tag{17}$$

This is the orbit equation in noncommutative Schwarzschild spacetime. If we use the following approximation for incomplete gamma function (see Appendix)

$$\gamma\Big(\frac{3}{2}, \frac{r^2}{4\theta}\Big)\mid_{\frac{r^2}{4\theta}\gg 1} \approx \frac{\sqrt{\pi}}{2} + \frac{1}{2}\frac{r}{\sqrt{\theta}}e^{\frac{-r^2}{4\theta}} \tag{18}$$

then we can write equation (17) as follows

$$\frac{d^2u}{d\varphi^2} + u = \frac{M}{h^2} + 3Mu^2 + \frac{M}{\sqrt{\pi}h^2\sqrt{\theta}}\frac{e^{-1/4\theta u^2}}{u}$$
$$-\frac{M}{2\sqrt{\pi}h^2\theta^{3/2}}\frac{e^{-1/4\theta u^2}}{u^3} + \frac{3Mu}{\sqrt{\pi}\sqrt{\theta}}e^{-1/4\theta u^2} - \frac{M}{2\sqrt{\pi}\theta^{3/2}}\frac{e^{-1/4\theta u^2}}{u} \tag{19}$$

The first two terms in the right hand side of equation (19) are the same as usual general relativity result[16], the other terms are noncommutative corrections. This equation has a complicated solution that can be simplified as follows (we set $c = 1$)

$$u \simeq \frac{M}{h^2}\left[1 + e\cos\varphi + 3\frac{M^2}{h^2}e\varphi\sin\varphi\right] + e\frac{M^2\sqrt{\theta}}{2\sqrt{\pi}h^4}\cos 2\varphi + \frac{M^3\sqrt{\theta}}{6\sqrt{\pi}h^5}e\varphi\sin\varphi \tag{20}$$

where

$$e = \frac{Ch^2}{M} \tag{21}$$

and C is a constant. The last two terms in (20) show the noncommutative correction and we defined eccentricity e as in commutative case. The general form of this result is in agreement with the one obtained by a different method in reference [9]. Since noncommutativity parameter is extremely small, the noncommutative corrections will be very small. However these corrections are important since they are related to the nature of spacetime structure at quantum gravity level. Using the relation

$$\cos[(1-\alpha)\varphi] = \cos\varphi + \alpha\frac{d}{d\alpha}\cos[(1-\alpha)\varphi]_{\alpha=0} = \cos\varphi + \alpha\varphi\sin\varphi$$

for small parameter α, one can rewrite equation (20) as follows

$$u \simeq \frac{M}{h^2}\left[1 + e\cos(\varphi(1-\alpha))\right] + e\frac{M^2\sqrt{\theta}}{2\sqrt{\pi}h^4}\cos 2\varphi \tag{22}$$

where($c = 1$)

$$\alpha = \frac{3M^2}{h^2} + \frac{M^2\sqrt{\theta}}{6\sqrt{\pi}h^3}. \tag{23}$$

So the period of the orbit is

$$\frac{2\pi}{1-\alpha} \simeq 2\pi(1+\alpha) = 2\pi(1 + \frac{3M^2}{h^2} + \frac{M^2\sqrt{\theta}}{6\sqrt{\pi}h^3}). \tag{24}$$

We have therefore found that, during each orbit of the particle, perihelion advances by an angle

$$\Delta\varphi = 2\pi\alpha = \frac{6\pi M^2}{h^2} + \frac{\pi M^2\sqrt{\theta}}{3\sqrt{\pi}h^3} \tag{25}$$

This contains an extra precession of the perihelion of the orbit due to space noncommutativity,

$$\left(\Delta\varphi\right)_{NC} = \frac{\pi M^2\sqrt{\theta}}{3\sqrt{\pi}h^3}. \tag{26}$$

In Newtonian formulation of orbital motion, angular momentum is given by $h^2 \approx GM(1-e^2)a$ where e is the eccentricity and a the semi-major axis of the orbit[15]. Using this relation and transforming (26) to non-relativistic units we find

$$\left(\Delta\varphi\right)_{NC} = \pi \frac{(GM\theta)^{1/2}}{3c\sqrt{\pi}[(1-e^2)a]^{3/2}}. \tag{27}$$

To have an estimation of this extra precession of perihelion, we consider the motion of Mercury around Sun. In this case we have $\frac{GM}{c^2} \simeq 1.48 \times 10^3\, m$, $e = 0.2056$, $a = 5.79 \times 10^{10}\, m$(see for example [15]) and $\theta \sim 10^{-72}\, m^2$ [13], we obtain

$$\left(\Delta\varphi\right)_{NC} \simeq \pi(0.55 \times 10^{-51})\ \ radians/orbit = 0.355 \times 10^{-45}\ \ arcseconds/orbit.$$

Mercury orbits once every 88 days, so we find

$$\left(\Delta\varphi\right)_{NC} \simeq 0.14722 \times 10^{-42}\ \ arcseconds/century.$$

We next consider the case of null geodesics to obtain effect of space noncommutativity on the light deflection in Schwarzschild geometry. This can be done by changing the right hand side of equation (7) to zero

$$2K = \left(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\right)\dot{t}^2 - \left(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\right)^{-1}\dot{r}^2 - r^2\dot{\vartheta}^2 - r^2\sin^2(\vartheta)\dot{\varphi}^2 = 0. \tag{28}$$

Using the same method as previous part, we reach at the following second order differential equation for the trajectory of a light ray in noncommutative Schwarzschild geometry

$$\frac{d^2u}{d\varphi^2} + u = \frac{6Mu^2}{\sqrt{\pi}}\gamma - \frac{M}{2\sqrt{\pi}\theta^{3/2}}\frac{e^{-1/4\theta u^2}}{u}. \tag{29}$$

Again using approximation (18) we can rewrite this equation as

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2 + \frac{3Mu}{\sqrt{\pi}\sqrt{\theta}}e^{-1/4\theta u^2} - \frac{M}{2\sqrt{\pi}\theta^{3/2}}\frac{e^{-1/4\theta u^2}}{u} \tag{30}$$

where the last two terms in the right hand side are the noncommutative corrections. The solution of this equation can be written in the following form

$$u = u_0 + u_1 \tag{31}$$

where u_0 is the special relativistic solution,

$$u_0 = \frac{1}{D}\sin\varphi, \tag{32}$$

and u_1 has the following form

$$u_1 \approx c_1 \cos\varphi + c_2 \sin\varphi + \frac{3}{2}\frac{M\sqrt{\theta}[\ln(\sin\varphi)\sin\varphi - \varphi\cos\varphi]}{D^3\sqrt{\pi}}. \tag{33}$$

D is a constant with dimension of length. Therefore, the general solution is

$$u \approx \frac{\sin\varphi}{D} + c_1 \cos\varphi + c_2 \sin\varphi + \frac{3}{2}\frac{M\sqrt{\theta}[\ln(\sin\varphi)\sin\varphi - \varphi\cos\varphi]}{D^3\sqrt{\pi}} \tag{34}$$

In a commutative space one can find the solution for u as[16]

$$u \simeq \frac{\sin\varphi}{D} + \frac{M(1+\cos\varphi+\cos^2\varphi)}{D^2}. \tag{35}$$

Therefore we can write u in noncommutative space as follows

$$u \simeq \frac{\sin\varphi}{D} + \frac{M(1+\cos\varphi+\cos^2\varphi)}{D^2} + \frac{3}{2}\frac{M\sqrt{\theta}[\ln(\sin\varphi)\sin\varphi - \varphi\cos\varphi]}{D^3\sqrt{\pi}} \tag{36}$$

The last term in this relation is the correction due to space noncommutativity. We can now find the angle of deflection for a light ray using equation (36). If we take the values for φ to be $-\epsilon_1$ and $\pi+\epsilon_2$ when $r \to \infty$(see for example [16]), then using small angle approximation for ϵ_1 and ϵ_2, we find the angle of deflection, δ, as follows

$$\delta = \epsilon_1 + \epsilon_2 = \frac{4M}{D}\Big[1 + \frac{3M\sqrt{\theta}}{2D^2\sqrt{\pi}}\Big] \tag{37}$$

Again, the second term is the correction due to space noncommutativity. Using this relation we try to find a numerical estimation for noncommutativity effect on the deflection of light ray. In non-relativistic units the contribution of space noncommutativity to the angle of deflection becomes

$$\Big(\delta\Big)_{NC} = \frac{6GM}{c^2D^3}\frac{GM\sqrt{\theta}}{c^2\sqrt{\pi}}. \tag{38}$$

Since $D = 6.66 \times 10^8 m$ (mean radius of sun), for a light ray just grazing the sun, the approximate value of the noncommutative correction becomes

$$\Big(\delta\Big)_{NC} \simeq 0.25 \times 10^{-55}.$$

Although this correction is very small, but it is important since it contains information about the nature of spacetime at quantum gravity level. Note that the value of this term is dependent on the value of M, the mass of central object. So for the case of very large masses such as very massive stars it may be large enough.

From another view point, equation (37) could be used to find a limit on θ using observational data of deflection of light rays around sun.

4 Stability of Circular Orbits

For noncommutative Schwarzschild line element as given by equation (4), there exist two Killing vectors associated with energy and the angular momentum . In the equatorial plane , $\vartheta = \frac{\pi}{2}$, the Killing vector associated with energy is ∂_t or

$$K_\mu = \Big([1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)],\, 0,\, 0,\, 0\Big) \tag{39}$$

and for the angular momentum the Killing vector is ∂_φ or

$$L_\mu = \Big(0,\, 0,\, 0,\, -r^2 sin^2\vartheta\Big). \tag{40}$$

So, along the geodesics, the two corresponding conserved quantities are

$$\Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\Big)\frac{dt}{d\lambda} = E \tag{41}$$

and

$$r^2\frac{d\varphi}{d\lambda} = L \tag{42}$$

respectively, where E and L are energy and angular momentum of the particle per its unit mass. In the equatorial plane, $\vartheta = \frac{\pi}{2}$ and using equations (9) and (41) we find

$$E^2 - \Big(\frac{dr}{d\lambda}\Big)^2 + \Big(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\Big)\Big(\frac{L^2}{r^2} + 1\Big) = 0 \tag{43}$$

or

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + V(r) = \frac{1}{2}E^2 \tag{44}$$

where we have defined

$$V(r) = \frac{2M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta) + \frac{2ML^2}{r^3\sqrt{\pi}}\gamma(3/2, r^2/4\theta) - \frac{L^2}{2r^2} - \frac{1}{2} \tag{45}$$

which is the *Effective Potential* in this noncommutative Schwarzschild spacetime. This noncommutative effective potential has been shown in figure 1 in comparison with commutative Schwarzschild case. The divergency around the origin is a manifestation of the existence of minimal length scale which prevents to probe distances smaller than a fundamental distance, for instance, Planck length. The particle could have a circular orbit at r_c if

$$\Big(\frac{\partial V}{\partial r}\Big)_{r=r_c} = 0. \tag{46}$$

Applying this to the effective potential (45) gives the equation which determines the radius of circular orbits

$$\frac{-2M}{r_c^2\sqrt{\pi}}\gamma(3/2,r^2/4\theta)+\frac{r_cM}{2\theta^{3/2}\sqrt{\pi}}e^{-r_c^2/4\theta}-\frac{6ML^2}{r_c^4\sqrt{\pi}}\gamma(3/2,r^2/4\theta)+\frac{ML^2}{2r_c\theta^{3/2}\sqrt{\pi}}e^{-r_c^2/4\theta}+\frac{L^2}{r_c^3}=0 \tag{47}$$

so the condition for the stability of the circular orbits is

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_{r=r_c}=\frac{4M}{r_c^3\sqrt{\pi}}\gamma(3/2,r^2/4\theta)+\frac{M}{4\theta^{3/2}\sqrt{\pi}}e^{-r_c^2/4\theta}-\frac{M(L^2+r_c^2)}{4\theta^{5/2}\sqrt{\pi}}e^{-r_c^2/4\theta}$$
$$+\frac{24ML^2}{r_c^5\sqrt{\pi}}\gamma(3/2,r^2/4\theta)-\frac{5ML^2}{4r_c^2\theta^{3/2}\sqrt{\pi}}e^{-r_c^2/4\theta}-\frac{3L^2}{r_c^4}\geq 0 \tag{48}$$

combining (47) and (48) we find

$$\frac{-2M}{r_c^3\sqrt{\pi}}\gamma(3/2,r^2/4\theta)+\frac{7M}{4\theta^{3/2}\sqrt{\pi}}e^{-r_c^2/4\theta}-\frac{M(L^2+r_c^2)}{4\theta^{5/2}\sqrt{\pi}}e^{-r_c^2/4\theta}$$
$$+\frac{6ML^2}{r_c^5\sqrt{\pi}}\gamma(3/2,r^2/4\theta)+\frac{ML^2}{4r_c^2\theta^{3/2}\sqrt{\pi}}e^{-r_c^2/4\theta}\geq 0 \tag{49}$$

This is a complicated relation with no analytical solution for r_c. Instead, we have depicted the left hand side of this relation in terms of radius. The result is shown in figure 2. In Newtonian mechanics the circular orbits are stable if $r_c \geq 3GM$. In commutative Schwarzschild geometry this orbits are stable when $r_c \geq 6GM$ [15]. Now we see that space noncommutativity increases the radius of stable circular orbits. As figure 5 shows, in noncommutative Schwarzschild spacetime the condition for stability of circular orbits is given by $r_c > 6.27\,GM$. So, the space noncommutativity increases the radius of stable circular orbits and this is a manifestation of smeared picture of objects in noncommutative geometry.

5 Summary and Conclusion

In this paper we have studied the effects of space noncommutativity on the orbits of particles in noncommutative Schwarzschild spacetime. The effects of space noncommutativity is so that it is not necessary to change the Einstein tensor part of the field equation, and one can argue that the noncommutative effects can be implemented acting only on the matter part of Einstein's equations. Using this picture we have calculated the noncommutative orbital motion of test particle in noncommutative Schwarzschild spacetime. An extra precession of the perihelion of the orbit due to space noncommutativity has been

calculated and its numerical value is estimated using observational data of Mercury. Although this noncommutative effect is very small, it is important since reflect the nature of spacetime structure at quantum gravity level. We have calculated the corrected angle of light deflection due to space noncommutativity in Schwarzschild spacetime. Noncommutative effective potential in Schwarzschild spacetime is calculated and its behavior is compared with commutative result. The stability of circular orbits in noncommutative Schwarzschild spacetime is discussed and radius of stable circular orbits is calculated. Radius of stable circular orbits increases due to space noncommutativity which is a manifestation of smeared picture of objects in noncommutative spacetime.

Appendix

Lower Incomplete Gamma Function

The lower incomplete gamma function is given by[13]

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt = a^{-1} x^a e^{-x} \, {}^1F^1(1; 1+a; x) = a^{-1} x^{a} {}^1F^1(a; 1+a; -x) \tag{50}$$

where ${}^1F^1(a;b;x)$is the confluent hypergeometric function of the first kind. Long and short distance behavior of lower incomplete gamma function are as follows

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) |_{\frac{r^2}{4\theta} \ll 1} \approx \frac{r^3}{12\sqrt{\theta^3}} \left(1 - \frac{7}{20} \frac{r^2}{\theta}\right) \tag{51}$$

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) |_{\frac{r^2}{4\theta} \gg 1} \approx \frac{\sqrt{\pi}}{2} + \frac{1}{2} \frac{r}{\sqrt{\theta}} e^{\frac{-r^2}{4\theta}}. \tag{52}$$

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References

[1] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977-1029

[2] R. J. Szabo, *Phys. Rept.* **378** (2003) 207-299

[3] N. Seiberg and E. Witten, *JHEP* **9909** (1999) 032

[4] A. Connes and M. Marcolli, arXiv:math.QA/0601054
A. Connes, *J. Math. Phys.* **41** (2000) 3832-3866

[5] A. Konechny and A. Schwarz, *Phys. Rept.* **360** (2002) 353-465;

[6] M. Chaichian *et al*, *Eur. Phys. J. C* **29** (2003) 413-432

[7] A. Micu and M.M. Sheikh-Jabbari, *JHEP* **0101** (2001) 025

[8] S. Benczik et al, *Phys. Rev. D* **66** (2002) 026003

[9] B. Mirza and M. Dehghani, *Commun. Theor. Phys.* **42** (2004) 183-184

[10] J.M Romero and J. D Vergara, *Mod. Phys. Lett. A* **18** (2003) 1673-1680

[11] K. Nozari and S. Akhshabi, arXiv:gr-qc/0608076

[12] A. Smailagic and E. Spallucci, *J. Phys. A* **36** (2003) L467
A. Smailagic and E. Spallucci, *J. Phys. A* **36** (2003) L517

[13]
S. Ansoldi, P. Nicolini, A. Smailagic and E. Spallucci, *Phys. Lett. B* **645** (2007) 261-266
P. Nicolini, A. Smailagic and E. Spalluci, *Phys. Lett. B* **632** (2006) 547-551
E. Spallucci, A. Smailagic and P. Nicolini, *Phys. Rev. D* **73** (2006) 084004
P. Nicolini, *J. Phys. A* **38** (2005) L631-L638

[14] T. G. Rizzo, *JHEP* **09** (2006) 021

[15] S. M. Carroll, *An Introduction to General Relativity: Spacetime and Geometry*, Addison Wesley, 2004

[16] R. d'Inverno, *Introducing Einstein's Relativity*, Oxford University Press, USA (1992)

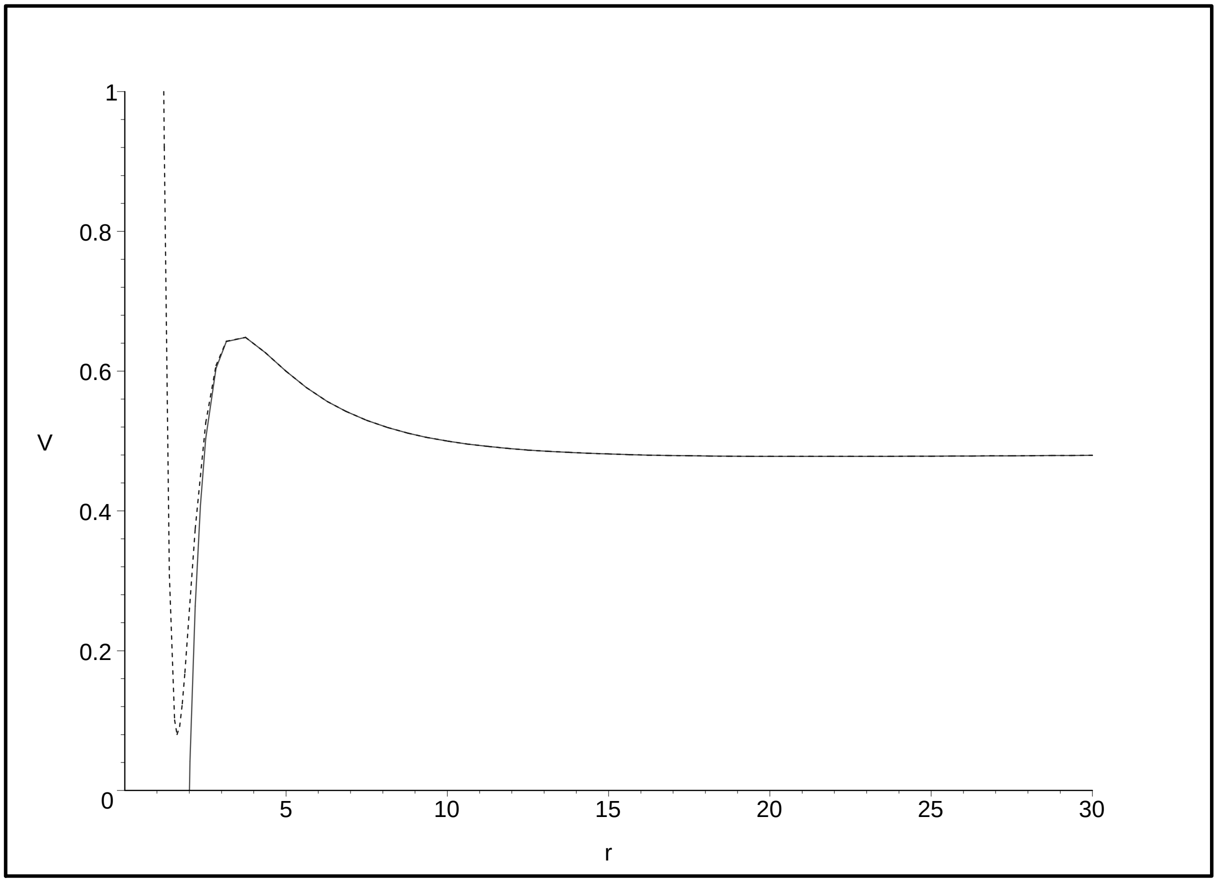

Figure 1: Effective potential in noncommutative Schwarzschild spacetime(dotted curve) in comparison with the commutative case(solid line). Divergent behavior of effective potential around origin in noncommutative case is a manifestation of existence of minimal length scale which prevents one to probe distances smaller than this minimal length. This minimal observable length is of the order of Planck length.

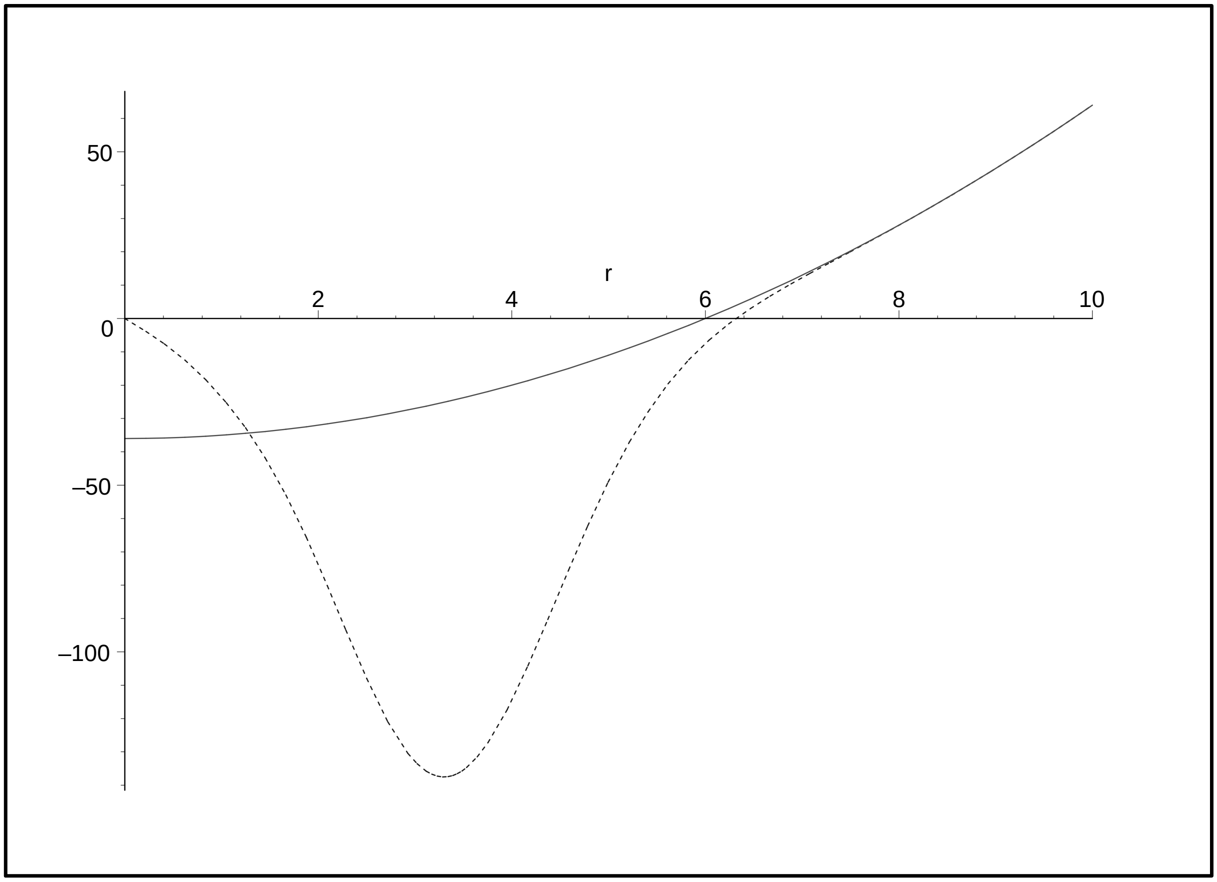

Figure 2: The condition for stability of circular orbits of particles in Schwarzschild spacetime: In commutative case the condition for stability is given by $r \geq 6GM$. In noncommutative situation the condition for stability of circular orbits is given by relation (49) which is shown by lower(doted) curve. In this case the condition of stability is $r \geq 6.27\,GM$